IRREDULAR SMOOTHING AND THE NUMBER OF REIDEMEISTER MOVES

CHUICHIRO HAYASHI AND MIWA HAYASHI

ABSTRACT. In the previous paper [6], we consider a link diagram invariant of Hass and Nowik type using regular smoothing and unknotting number, to estimate the number of Reidemeister moves needed for unlinking. In this paper, we introduce a new link diagram invariant using irregular smoothing, and give an example of a knot diagram of the unknot for which the new invariant gives a better estimation than the old one.

Mathematics Subject Classification 2010: 57M25.
Keywords: link diagram, Reidemeister move, irregular smoothing, unknotting number.

1. INTRODUCTION

In this paper, we regard a knot as a link with one component, and assume that link diagrams are in the 2-sphere. Links and link diagrams are assumed to be oriented unless otherwise specified. A *Reidemeister move* is a local move of a link diagram as in Figure 1. An RI (resp. II) move creates or deletes a monogon face (resp. a bigon face). An RII move is called *matched* and *unmatched* according to the orientations of the edges of the bigon as shown in Figure 2. An RIII move is performed on a 3-gon face, deleting it and creating a new one. Any such move does not change the link type. As Alexander and Briggs [1] and Reidemeister [15] showed, for any pair of diagrams D_1, D_2 which represent the same link type, there is a finite sequence of Reidemeister moves which deforms D_1 to D_2.

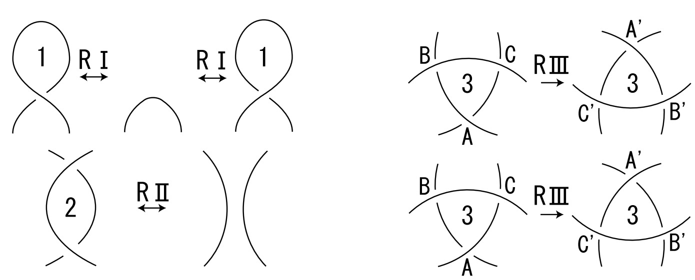

FIGURE 1.

Necessity of Reidemeister moves of type II and III is studied in [14], [12] and [3]. There are several studies of lower bounds for the number of Reidemeister moves connecting two

The first author is partially supported by Grant-in-Aid for Scientific Research (No. 22540101), Ministry of Education, Science, Sports and Technology, Japan.

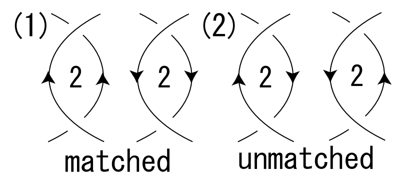

FIGURE 2.

knot diagrams of the same knot. See [17], [4], [2], [8], [9], [5], [7] and [6]. Hass and Nowik introduced in [8] a certain knot diagram invariant I_ϕ by using (regular) smoothing and an arbitrary link invariant ϕ. In [9], setting ϕ to be the linking number, they showed that a certain infinite sequence of diagrams of the unknot requires quadratic number of Reidemeister moves with respect to the number of crossings for being unknotted. In the previous paper [6], we consider a link diagram invariant of Hass and Nowik type using the unknotting number. Note that we obtain a diagram of a knot rather than a multi-component link when we perform a smooting at a crossing between ditinct components of a 2-component link. We have considered the unknotting number rather than the linking number to deal with this situation. This also allows us to perform an irregular smoothing on a knot diagram. In this paper, we introduce a new link diagram invariant using irregular smoothing, and give an example of a diagram of the unknot for which the new invariant gives a better estimation of the number of Reidemeister moves than the old one.

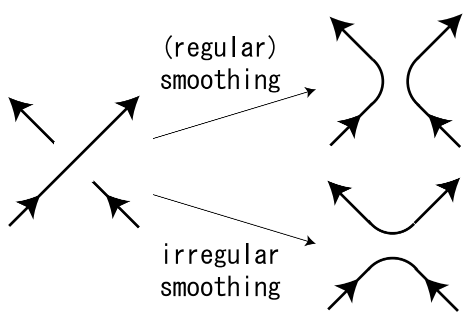

FIGURE 3.

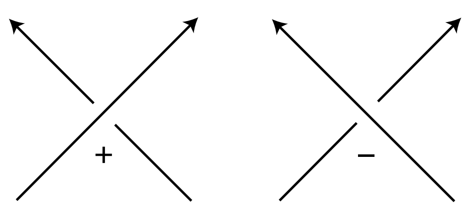

FIGURE 4.

An *irregular smoothing* at a crossing of a link diagram is an operation as in Figure 3. We regard the resulting link digram to be unoriented. For a link K of m components, the *unknotting number* $u(K)$ of K is the X-Gordian distance between K and the trivial link of m components. (Precise descriptions of the definitions of the irregular smoothing and the unknotting number are given in Section 2.) Then we define the new link diagram invariant $iu_{S,T}$ as below. Let L be an oriented n-component link with its components numbered as $L_1, L_2, \cdots L_n$. Let D be a diagram of L in the 2-sphere. For a crossing x of D, let D_x (resp. $\check{D}_x$) denote the link obtained from D by performing a regular smoothing operation (resp. an irregular smoothing operation) at x. Note that D_x ($\check{D}_x$) is a link rather than a diagram.

The link D_x (resp. $\check{D}_x$) has $n+1$ (resp. n) components when x is a crossing between subarcs of the same component, and $n-1$ (resp. $n-1$) components when x is a crossing between subarcs of distinct components. Let S and T be symmetric matrices of order n with each element s_{ij}, t_{ij} ($1 \leq i, j \leq n$) is equal to -1, 0 or $+1$ such that $t_{ij} = 0$ if and only if $s_{ij} = 0$. We set $\mathcal{C}(D)$ (resp. $\check{\mathcal{C}}(D)$) to be the set of all the crossings of D between subarcs of components L_i and L_j with the (i, j)-element $s_{ij} = +1$ (resp. $s_{ij} = -1$). Then we set

$$iu_{S,T}(D) = \sum_{x \in \mathcal{C}(D)} t(x) \cdot \mathrm{sign}(x) \cdot |\Delta u(D_x)| + \sum_{x \in \check{\mathcal{C}}(D)} t(x) \cdot \mathrm{sign}(x) \cdot |\Delta u(\check{D}_x)|,$$

where $\Delta u(D_x)$ (resp. $\Delta u(\check{D}_x)$) is the difference between the unknotting numbers of D_x and L (resp. $\check{D}_x$ and L), i.e., $\Delta u(D_x) = u(D_x) - u(L)$ (resp. $\Delta u(\check{D}_x) = u(\check{D}_x) - u(L)$), and $t(x)$ denotes the (i, j)-element of T such that x is a crossing between a subarc of the component L_i and that of L_j. The sign of a crossing $\mathrm{sign}(x)$ is defined as in Figure 4 as usual. We set $iu_S(D) = 0$ for a diagram D with no crossings.

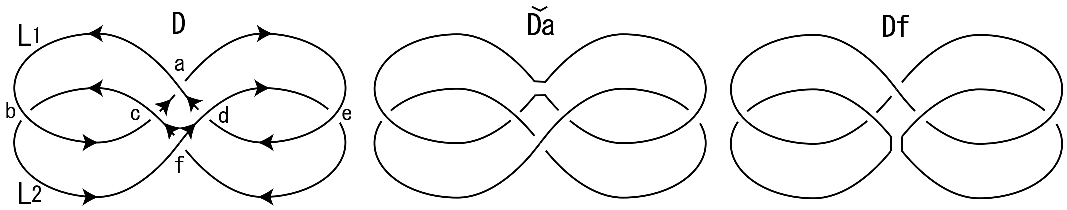

FIGURE 5.

For example, let D be a link diagram in Figure 5. The link has two components L_1 and L_2 and is trivial. We set $S = T = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. Then the irregular smoothing at the crossing a yields $\check{D}_a$ which is the $(2,4)$-torus link and has the unknotting number $u(\check{D}_a) = 2$. Since $t_{11} = -1$ and $\mathrm{sign}(a) = -1$, the crossing a contributes by $(-1)(-1)|2-0|$ to $iu_{S,T}(D)$. The crossings b through e do not contribute to $iu_{S,T}(D)$ because $s_{12} = 0$. The link D_f obtained from D by the regular smoothing at f is a composite of two Hopf links, and has the unknotting number $u(D_f) = 2$. Since $t_{22} = +1$ and $\mathrm{sign}(f) = +1$, the crossing f contributes by $(+1)(+1)|2-0|$ to $iu_{S,T}(D)$. Hence we have $iu_{S,T}(D) = (-1)(-1)|2-0| + (+1)(+1)|2-0| = 4$.

Remark 1.1. If we replace $|\Delta u(D_x)|$ and $|\Delta u(\check{D}_x)|$ with $\Delta u(D_x)$ and $\Delta u(\check{D}_x)$, in the definition of $iu_{S,T}(D)$, we obtain another link diagram invariant as below.

$$\tilde{iu}_{S,T}(D) = \sum_{x \in \mathcal{C}(D)} t(x) \cdot \mathrm{sign}(x) \cdot \Delta u(D_x) + \sum_{x \in \check{\mathcal{C}}(D)} t(x) \cdot \mathrm{sign}(x) \cdot \Delta u(\check{D}_x)$$

All the results about $iu_{S,T}(D)$ in this paper hold also for $\tilde{iu}_{S,T}(D)$.

Theorem 1.2. *The link diagram invariant $iu_{S,T}(D)$ does not change under an RI move, and changes at most by one under an RII move, and at most by two under an RIII move.*

The above theorem is proved in Section 2.

Corollary 1.3. *Let D_1 and D_2 be link diagrams of the same oriented link. We need at least $\lceil |iu_{S,T}(D_1) - iu_{S,T}(D_2)|/2 \rceil$ RII and RIII moves to deform D_1 to D_2 by a sequence of Reidemeister moves, where $\lceil x \rceil$ denotes the smallest integer larger than or equal to x. In particular, when D_2 is a link diagram with $\mathcal{C}(D_2) \cup \check{\mathcal{C}}(D_2) = \emptyset$, we need at leat $\lceil |iu_{S,T}(D_1)|/2 \rceil$ RII and RIII moves.*

Note that, for estimation of the unknotting number, we can use the signature and the nullity (see Theorem 10.1 in [13] and Corollary 3.21 in [10]) or the sum of the absolute values of linking numbers over all pairs of components.

For example, the diagram D in Figure 5 needs at least $\lceil iu_{S,T}(D)/2 \rceil = \lceil 4/2 \rceil = 2$ RII and RIII moves to be deformed so that it has no crossing between subarcs of the same component.

In general, for a link diagram D, the sum of the signs of all the crossings is called the *writhe* and denoted by $w(D)$. It does not change under an RII or RIII move but increases (resp. decreases) by one under an RI move creating a positive (resp. negative) crossing. Set ${}_{\epsilon,\delta}iu_{S,T}(D) = iu_{S,T}(D) + \epsilon(\frac{1}{2}c(D) + \delta\frac{3}{2}w(D))$ for a link diagram D, where $\epsilon = \pm 1$, $\delta = \pm 1$ and $c(D)$ denotes the number of crossings of D. Then we have the next corollary.

Corollary 1.4. *The link diagram invariant ${}_{\epsilon,+1}iu_{S,T}(D)$ (resp. ${}_{\epsilon,-1}iu_{S,T}(D)$) increases by 2ϵ under an RI move creating a positive (resp. negative) crossing, decreases by ϵ under an RI move creating a negative (resp. positive) crossing. ${}_{\epsilon,\delta}iu_{S,T}(D)$ changes at most by 2 under an RII or RIII move.*

Let D_1 and D_2 be link diagrams of the same oriented link. We need at least $\lceil |{}_{\epsilon,\delta}iu_{S,T}(D_1) - {}_{\epsilon,\delta}iu_{S,T}(D_2)|/2 \rceil$ Reidemeister moves to deform D_1 to D_2. In particular, when D_2 is a link diagram with $\mathcal{C}(D_2) \cup \check{\mathcal{C}}(D_2) = \emptyset$, we need at least $\lceil |{}_{\epsilon,\delta}iu_{S,T}(D_1)|/2 \rceil$ Reidemeister moves.

For a knot diagram D, we consider only $iu_{S,T}(D)$, ${}_{\epsilon,\delta}iu_{S,T}(D)$ with $T = (+1)$, and we denote them by $iu_S(D)$, ${}_{\epsilon,\delta}iu_S(D)$ for short.

Theorem 1.5. *For the diagram U of the unknot in Figure 7, which can be unknotted by the sequence of 7 Reidemeister moves in the figure, $g(I_{lk}(U)) = 4$, $max_{\epsilon=\pm1,\delta=\pm1} \lceil |{}_{\epsilon,\delta}iu_{(+1)}(U)/2| \rceil = 6$ and $\lceil |{}_{+1,+1}iu_{(-1)}(U)/2| \rceil = 7$, where $g(I_{lk})$ denotes Hass and Nowik's invariant (see Section 3 for the definition).*

Theorem 1.6. *There is a knot diagram which admits an RIII move under which $iu_{(+1)}$ does not change and $iu_{(-1)}$ changes.*

Because $\epsilon(\frac{1}{2}c(D)+\delta\frac{3}{2}w(D))$ is unchanged under an RIII move, the above theorem holds also for ${}_{\epsilon,\delta}iu_{(+1)}$ and ${}_{\epsilon,\delta}iu_{(-1)}$. Note that the Hass and Nowik's invariant $g(I_{lk})$ always changes by one under an RIII move.

We prove Theorems 1.5 and 1.6 in Section 3.

2. Change of the link diagram invariant under a Reidemeister move

Let D be an oriented link diagram in the 2-sphere, and p a crossing of D. A (regular) *smoothing* at p is an operation which yields a new link diagram D'_p from D as below. We first cut the link at the two preimage points of p. Then we obtain the four endpionts. We paste the four short subarcs of the link near the endpoints in the way other than the original one so that their orientations are connected consistently. See Figure 3. If we paste the four short subarcs so that their orientations are inconsistent, we obtain another link diagram, which we denote by $\check{D}'_p$. We consider $\check{D}'_p$ unoriented. We call this operation deforming D to $\check{D}'_p$ an *irregular smoothing* at p.

The *trivial n-component link* is the link which has n components and bounds a disjoint union of n disks. The trivial n-component link admits a link diagram with no crossings, which we call a *trivial diagram.*

Let L be a link with n components, and D a diagram of L. We call a sequence of Reidemeister moves and crossing changes on D an *X-unknotting sequence* in this paragraph if it deforms D into a (possibly non-trivial) diagram of the trivial n-component link. The *length* of an X-unknotting sequence is the number of crossing changes in it. The minimum length among all the X-uknotting sequences on D is called the *uknotting number* of L. We denote it by $u(L)$.

Then we define the link diagram invariants $iu_{S,T}$, ${}_{\epsilon,\delta}iu_{S,T}$ as in Section 1.

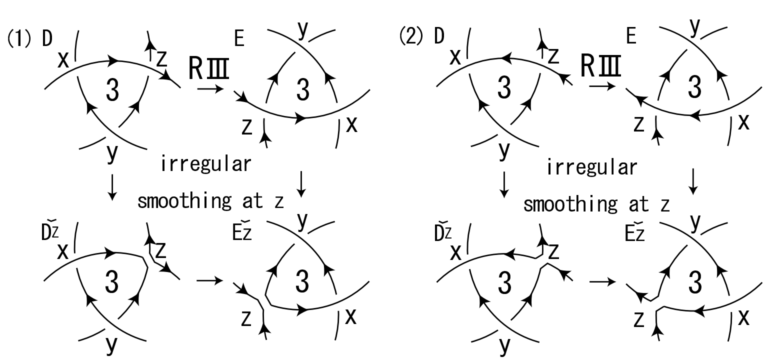

FIGURE 6.

Proof of Theorem 1.2. The proof is very similar to the arguments in Section 2 in [8] and in Section 2 in [6]. Let D, E be link diagrams of a link L such that E is obtained from D by a Reidemeister move.

First, we suppose that E is obtained from D by an RI move creating a crossing a. Let L_i be a component of L such that a is a crossing between two subarcs of L_i. When the element s_{ii} of the symmetric matrix S is $+1$, the same argument as in [6] works, and we omit it. We consider the case where $s_{ii} = -1$. Then the link $\check{E}_a$ obtained from E by an irregular smoothing at a is of the same link type as the original link L, and hence $u(\check{E}_a) = u(L)$. Then the contribution of a to $iu_{S,T}(E)$ is $\pm(u(\check{E}_a) - u(L)) = 0$. The contribution of any other crossing x to $iu_{S,T}$ is unchanged since the RI move shows that D_x and E_x are the same link. Thus the RI move does not change $iu_{S,T}$, i.e., $iu_{S,T}(D) = iu_{S,T}(E)$. When $s_{ii} = 0$, the crossing a does not contributes to $iu_{S,T}(E)$, and $iu_{S,T}$ does not change.

We consider the case where E is obtained from D by an RII move creating a bigon face. Let x and y be the positive and negative crossings at the corners of the bigon, and L_i, L_j components of L such that the edges of the bigon are a subarc of L_i and a subarc of L_j. It is possible that $i = j$. When $s_{ij} = +1$, the proof is the same as in Section 2 in [6], and we omit it. When $s_{ij} = 0$, it is clear that $iu_{S,T}$ does not change. Hence we assume that $s_{ij} = -1$. If the RII move is matched, then $\check{E}_x$ and $\check{E}_y$ are the same link. Hence $u(\check{E}_x) = u(\check{E}_y)$ and $|iu_{S,T}(E) - iu_{S,T}(D)| = |t_{ij}(|u(\check{E}_x) - u(L)| - |u(\check{E}_y) - u(L)|)| = 0$. If the RII move is unmatched, then a crossing change on $\check{E}_x$ yields $\check{E}_y$, and hence their unknotting numbers differ by at most one, i.e., $|u(\check{E}_x) - u(\check{E}_y)| \leq 1$. Hence $|iu_{S,T}(E) - iu_{S,T}(D)| = ||u(\check{E}_x) - u(L)| - |u(\check{E}_y) - u(L)|| \leq |(u(\check{E}_x) - u(L)) - (u(\check{E}_y) - u(L))| = |u(\check{E}_x) - u(\check{E}_y)| \leq 1$.

We consider the case where E is obtained from D by an RIII move. See Figure 6 for typical examples. In D and E, let x be the crossing between the top and the middle strands of the trigonal face where the RIII move is applied. Let L_i and L_j be the components of L such that L_i contains the top strand and L_j contains the middle strand. Then D_x and E_x are the same link when $s_{ij} = +1$. Also, irregular smoothing operations on D and E at x yields the same link $\check{D}_x$ and $\check{E}_x$ when $s_{ij} = -1$. Hence the contribution of x to $iu_{S,T}$ is unchanged under the RIII move. The same is true for the crossing y between the bottom and the middle strands. Let z be the crossing between the top strand in L_m and the bottom strand in L_n. The proof is the same as in Section 2 in [6]. We recall it in the case where $s_{mn} = -1$. $\check{D}_z$ and $\check{E}_z$ differ by Reidemeister moves and at most two crossing changes, and hence $|u(\check{E}_z) - u(\check{D}_z)| \leq 2$. Thus $|iu_{S,T}(E) - iu_{S,T}(D)| = ||u(\check{E}_z) - u(L)| - |u(\check{D}_z) - u(L)|| \leq |(u(\check{E}_z) - u(L)) - (u(\check{D}_z) - u(L))| = |u(\check{E}_z) - u(\check{D}_z)| \leq 2$. □

3. Examples

In this section, we prove Theorems 1.5 and 1.6.

We first recall the definition of Hass and Nowik's invariant $g(I_{lk})$ defined in [8] and [9]. Let $\mathbb{G}_{\mathbb{Z}}$ be the free abelian group with basis $\{X_n, Y_n\}_{n \in \mathbb{Z}}$. The invariant I_{lk} assigns an element of $\mathbb{G}_{\mathbb{Z}}$ to a knot diagram. Let D be an oriented knot diagram. Then I_{lk} is defined by the formula $I_{lk}(D) = \sum_{p \in \mathcal{C}^+(D)} X_{lk(D_p)} + \sum_{m \in \mathcal{C}^-(D)} Y_{lk(D_m)}$, where $\mathcal{C}^+(D)$ (resp. $\mathcal{C}^-(D)$)

is the set of all the positive (resp. negative) crossings of D and lk is the linking number. Let $g : \mathbb{G}_{\mathbb{Z}} \to \mathbb{Z}$ be the homomorphism defined by $g(X_n) = |n| + 1$ and $g(Y_n) = -|n| - 1$. Then the numerical invariant $g(I_{lk}(D))$ of a knot diagram D changes at most by one under a Reidemeister move. This invariant can be decomposed as $g(I_{lk}(D)) = g_0(I_{lk}(D)) + w(D)$, where $w(D)$ is the writhe and the homomorphism $g_0 : \mathbb{G}_{\mathbb{Z}} \to \mathbb{Z}$ is defined by $g_0(X_n) = |n|$ and $g_0(Y_n) = -|n|$. It can be easily seen that $g_0(I_{lk}(D))$ does not change under an RI move and the change of $g_0(I_{lk}(D))$ under an RII or RIII move is the same as that of $g(I_{lk}(D))$.

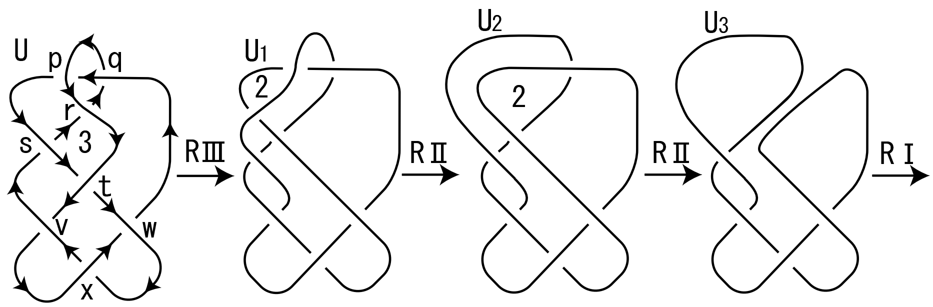

FIGURE 7.

Proof of Theorem 1.5. We consider the diagram U in Figure 7, which represents the trivial knot unknotted by the sequence of Reidemeister moves in Figure 7. We can unknot the last diagram by four RI moves.

For this diagram, $g_0(I_{lk}(U)) = 0$, $iu_{(+1)}(U)/2 = 1/2$ and $iu_{(-1)}(U)/2 = 3/2$.

In fact, U has 8 crossings p through t and v through x. We obtain the links in Figure 8 from U by a regular smoothing operation. Each of U_p, U_q and U_s is the $(2, 4)$-torus link with one component furnished the reversed orientation. Hence $|lk(U_p)| = |lk(U_q)| = |lk(U_s)| = 2$ and $u(U_p) = u(U_q) = u(U_s) = 2$. U_r is a composite of the twist knot 5_2 in the Rolfsen table [16] and the Hopf link. Hence $|lk(U_r)| = 1$ and $u(U_r) = 2$. Note that we need a crossing change at a crossing between distinct components for unlinking, and another crossing change at a crossing between subarcs of the twist knot component for unknot the twist knot. U_t is the Hopf link, and hence $|lk(U_t)| = 1$ and $u(U_t) = 1$. U_v, U_w and U_x are the trivial 2-component links, and $|lk| = u = 0$ for these links. Since $\operatorname{sign} p = \operatorname{sign} q = -1$, and $+1$ for the other crossings, $g_0(I_{lk}(U)) = -2 - 2 + 1 + 2 + 1 + 0 + 0 + 0 = 0$, and $iu_{(+1)}(U) = -2 - 2 + 2 + 2 + 1 + 0 + 0 + 0 = 1$.

We obtain the knots in Figure 9 from U by an irregular smoothing operation. $\check{U}_p$, $\check{U}_q$ and $\check{U}_t$ are the trefoil knots, and $u(\check{U}_p) = u(\check{U}_q) = u(\check{U}_t) = 1$. $\check{U}_r$ is the 7_4 knot, and $u(\check{U}_r) = 2$ by [11]. $\check{U}_s$ is the star knot 5_2, and $u(\check{U}_s) = 2$ (see [13]). $\check{U}_v$, $\check{U}_w$ and $\check{U}_x$ are the trivial knots, and $u(\check{U}_v) = u(\check{U}_w) = u(\check{U}_x) = 0$. Hence $iu_{(-1)}(U) = -1 - 1 + 2 + 2 + 1 + 0 + 0 + 0 = 3$.

Since $c(U) = 8$ and $w(D) = 4$, the expression $\epsilon(\frac{1}{2}c(D) + \delta\frac{3}{2}w(D))$ takes the maximum value 10 when $\epsilon = +1$ and $\delta = +1$. Because $iu_{(+1)}(U)$ and $iu_{(-1)}(U)$ are positive,

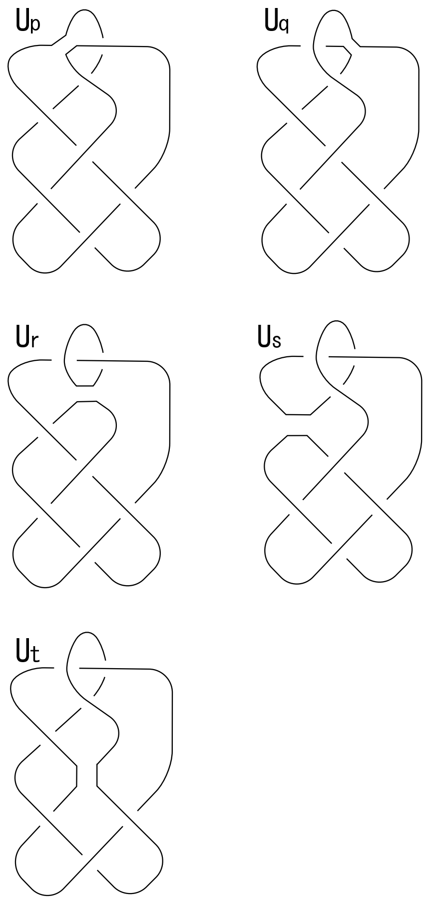

FIGURE 8.

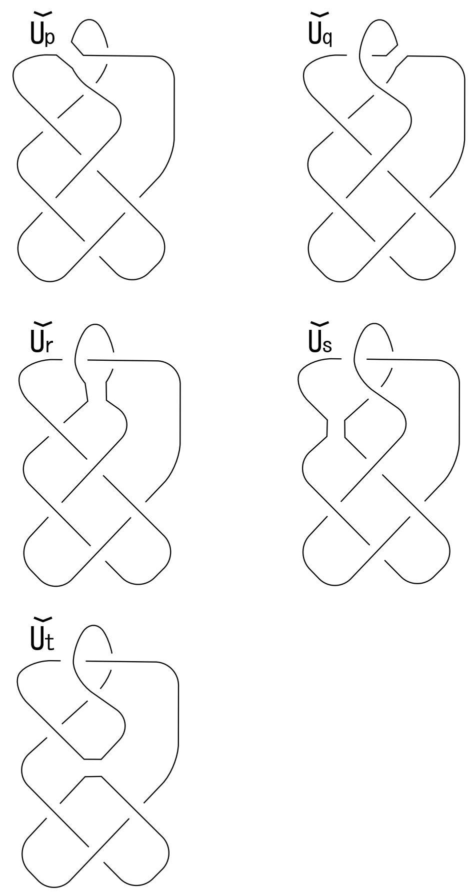

FIGURE 9.

$|{}_{\epsilon,\delta}iu_{(+1)}(U)|$ and $|{}_{\epsilon,\delta}iu_{(-1)}(U)|$ are the largest when $\epsilon = +1$ and $\delta = +1$. Then easy calculations show the theorem. □

Remark 3.1. The length of the unknotting sequence of Reidemeister moves in Figure 7 is 7. Without using ${}_{+1,+1}iu_{(-1)}(U)$, we can easily see that this is minimal as below. Since the writhe $w(U) = 4$, we need four RI moves for unknotting. These RI moves decrease the number of crossings of U by four. Because U has eight crossings, we need to delete four more crossings. Hence at least two more RI and RII moves are necessary for unknotting. We need at least one more Reidemeister move since U has neither a monogon face nor a bigon face on which we cay apply an RI or RII move to decrease the number of crossings.

$g(I_{lk})$ does not change under the second RII move in Figure 7 because the move is unmatched. Under the sequence of Reidemeister moves in Figure 7, $iu_{(+1)}$ varies $1 \to -1 \to 0 \to 0 \to 0 \to 0 \to 0 \to 0$, $iu_{(-1)}$ varies $3 \to 1 \to 1 \to 0 \to 0 \to 0 \to 0 \to 0$, ${}_{+1,+1}iu_{(+1)}$ varies $11 \to 9 \to 9 \to 8 \to 6 \to 4 \to 2 \to 0$, ${}_{+1,+1}iu_{(-1)}$ varies $13 \to 11 \to 10 \to 8 \to 6 \to 4 \to 2 \to 0$. So, ${}_{+1,+1}iu_{(-1)}$ is the most sensitive to each Reidemeister move in this case.

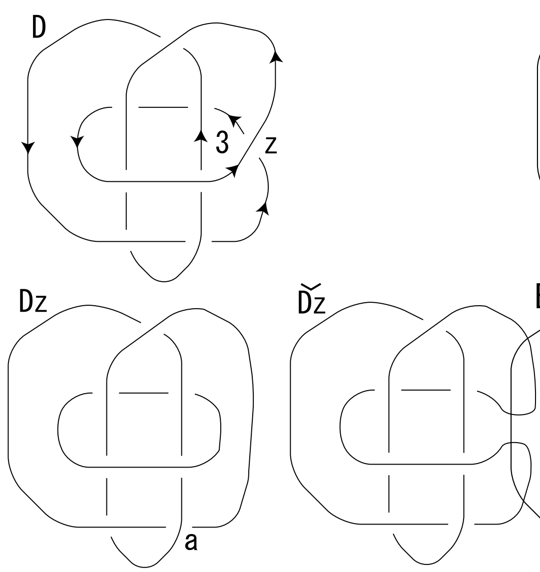

Figure 10.

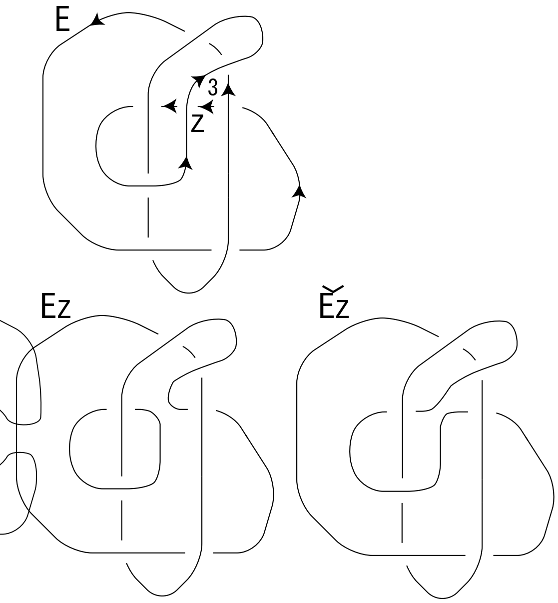

Figure 11.

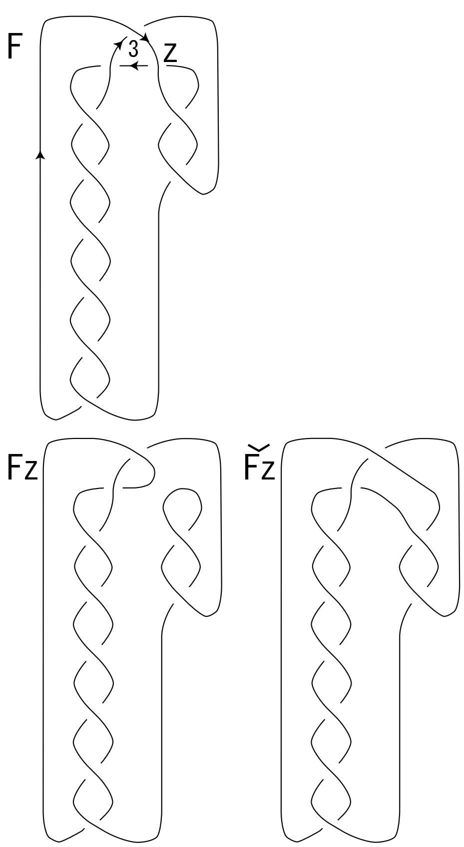

Figure 12.

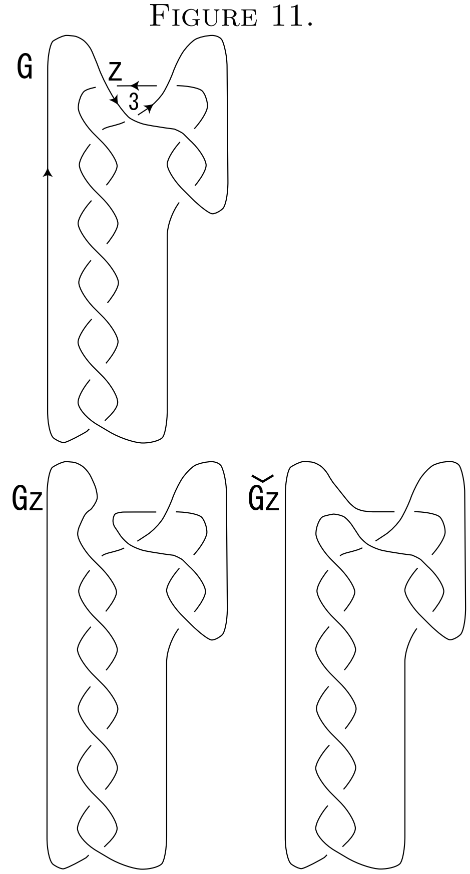

Figure 13.

Proof of Theorem 1.6. We give two examples: the first one is an RIII move on a diagram of the trivial knot in Figures 10 and 11 under which $iu_{(+1)}$ does not change and $iu_{(-1)}$ changes by one, and the second one is an RIII move on a diagram of a twist knot in Figures 12 and 13 under which $iu_{(+1)}$ does not change and $iu_{(-1)}$ changes by two. The RIII moves are performed at the trigonal faces labeled 3. In these diagrams, let z be the crossing between the top and the bottom strands of the trigonal face.

The diagram D in Figure 10 is deformed into the diagram E in Figure 11. As shown in the argument in the proof of Theorem 1.2 in Section 2, the changes of $iu_{(+1)}$ and $iu_{(-1)}$ are calculated as below.
$|iu_{(+1)}(E) - iu_{(+1)}(D)| = ||u(E_z) - u(L)| - |u(D_z) - u(L)|| = ||1-0| - |1-0|| = 0$
$|iu_{(-1)}(E) - iu_{(-1)}(D)| = ||u(\check{E}_z) - u(L)| - |u(\check{D}_z) - u(L)|| = ||1-0| - |0-0|| = 1$
Note that $u(D_z) = 1$ since a crossing change at the crossing a in Figure 11 yields the trivial 2-component link. $u(E_z) = 1$, $u(\check{D}_z) = 1$ and $u(\check{E}_z) = 0$ because E_z is the Hopf link, $\check{D}_z$ the twist knot 5_2 and $\check{E}_z$ the trivial knot.

The diagram F in Figure 12 is deformed into the diagram G in Figure 13. The changes of $iu_{(+1)}$ and $iu_{(-1)}$ are calculated as below.
$|iu_{(+1)}(G) - iu_{(+1)}(F)| = ||u(G_z) - u(L)| - |u(F_z) - u(L)|| = ||4-1| - |4-1|| = 0$
$|iu_{(-1)}(G) - iu_{(-1)}(F)| = ||u(\check{G}_z) - u(L)| - |u(\check{F}_z) - u(L)|| = ||1-1| - |3-1|| = 2$
Note that $u(\check{F}_z) = 3$ because $\check{F}_z$ is the knot 10_2 in the Rolfsen table [16] and its signature is -6 (see [13]). $u(G_z) = 4$ since G_z is the composite of the $(2,6)$-torus link and the trefoil knot. (We need at least 3 crossing change at crossings between distinct components to split the two components because $lk(G_z) = 3$, and at least one crossing change between subarcs of the trefoil knot component to unknot this component.) $u(F_z) = 4$ and $u(\check{G}_z) = 1$ because F_z is the $(2,8)$-torus link and $\check{G}_z$ the trefoil knot. □

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